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Non-commutative vs. Commutative Descriptions of D-brane BIons

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ABSTRACT

The $U(1)$ gauge theory on a D3-brane with non-commutative worldvolume is shown to admit BIon-like solutions which saturate a BPS bound on the energy. The mapping of these solutions to ordinary fields is found exactly, namely non-perturbatively in the non-commutativity parameters. The result is precisely an ordinary supersymmetric BIon in the presence of a background B-field. We argue that the result provides evidence in favour of the exact equivalence of the non-commutative and the ordinary descriptions of D-branes.

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1 Introduction and Conclusions

It has been recently appreciated [1, 2, 3] that the dynamics of D-branes in a constant background B-field admits two equivalent descriptions: either in terms of an 'ordinary' gauge theory, or in terms of a gauge theory on a *non-commutative* worldvolume. The two descriptions arise as a result of using two different regularizations for the open string worldsheet theory [3]. To provide evidence that both descriptions are equivalent *non-perturbatively* in the non-commutativity parameters is the purpose of this paper.

If Pauli-Villars regularization is used, the effective action for the massless open string fields enjoys an ordinary $U(N)$ gauge symmetry, where N is the number of overlapping branes. This gauge symmetry is therefore commutative in the case that one single D-brane is present, which will be the only case considered in the present paper. The (bosonic) massless fields are a gauge potential A_μ ($\mu = 0, \ldots, p$) and a number of scalars X^i ($i = p+1, \ldots, 9$). With Pauli-Villars regularization, the parameters entering the effective action are the closed string metric g, the closed string coupling constant g_s and the constant background B-field itself. Furthermore, the whole dependence of the effective action on B is accounted for by writing it in terms of the modified field strength $\mathcal{F} \equiv F + B^\star$, where $F = dA$ and the superscript '*' denotes the pull-back to the brane worldvolume. This is the combination which is invariant under the two $U(1)$ gauge symmetries which the open string σ-model enjoys at the classical level: a first one under which

$$\delta B = 0\,, \qquad \delta A = d\lambda\,, \tag{1.1}$$

and a second one under which

$$\delta B = d\Lambda\,, \qquad \delta A = -\Lambda^\star\,. \tag{1.2}$$

On the contrary, if point-splitting regularization is used, the gauge symmetry group of the effective action becomes non-commutative even in the case of one single D-brane. The theory is now naturally formulated as a gauge theory on a non-commutative worldvolume [3], namely a worldvolume whose coordinates σ^μ do not commute but satisfy

$$[\sigma^\mu, \sigma^\nu] \equiv \sigma^\mu * \sigma^\nu - \sigma^\nu * \sigma^\mu = i\theta^{\mu\nu}\,. \tag{1.3}$$

The antisymmetric matrix θ appearing above measures the non-commutativity of the theory and defines the so-called '$*$-product' as

$$(f * g)(\sigma) \equiv e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\chi^\nu}} f(\sigma+\xi)\, g(\sigma+\chi)\Big|_{\xi=\chi=0}\,. \tag{1.4}$$

The only parameters entering the effective action when point-splitting regularization is used are θ, an open string metric G and an open string coupling constant G_s, whose relation with the closed string parameters is [1]

$$\theta = (g+B)^{-1}_{(A)}\,, \qquad G = g - Bg^{-1}B\,, \qquad G_s = g_s\left(\frac{\det(g+B)}{\det g}\right)^{1/2}\,, \tag{1.5}$$

[1]We will set $2\pi\alpha' = 1$.

where (A) stands for the antisymmetric part of the matrix. All products of fields in the effective action are now $*$-products, and θ enters the action only through the definition of the $*$-product. The non-commutative field strength is defined as

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_{[\mu}\hat{A}_{\nu]} - i[\hat{A}_\mu, \hat{A}_\nu]\,, \\ [\hat{A}_\mu, \hat{A}_\nu] &= \hat{A}_\mu * \hat{A}_\nu - \hat{A}_\nu * \hat{A}_\mu\,, \end{aligned} \tag{1.6}$$

where $\hat{A}_\mu$ is the non-commutative gauge field. With this regularization, the gauge symmetry under which the effective action is invariant is

$$\delta\hat{A} = d\hat{\lambda} + i[\hat{\lambda}, \hat{A}]\,, \qquad \delta\hat{F} = i[\hat{\lambda}, \hat{F}]\,. \tag{1.7}$$

As explained in [3], since both the ordinary and the non-commutative descriptions arise from different regularizations of the same worldsheet theory, there should be a field redefinition which maps gauge orbits in one description into gauge orbits in the other. This requirement, plus locality to any finite order in θ, enabled the authors in [3] to establish the following system of differential equations: [2]

$$\begin{aligned} \delta\hat{A}_\mu = \delta\theta^{\alpha\beta}\,\frac{\partial\hat{A}_\mu}{\partial\theta^{\alpha\beta}} &= -\frac{1}{4}\,\delta\theta^{\alpha\beta}\left\{\hat{A}_\alpha, \partial_\beta\hat{A}_\mu + \hat{F}_{\beta\mu}\right\}\,, \\ \delta\hat{X}^i = \delta\theta^{\alpha\beta}\,\frac{\partial\hat{X}^i}{\partial\theta^{\alpha\beta}} &= -\frac{1}{4}\,\delta\theta^{\alpha\beta}\left\{\hat{A}_\alpha, \partial_\beta\hat{X}^i + D_\beta\hat{X}^i\right\}\,, \end{aligned} \tag{1.8}$$

where $\{f, g\} \equiv f * g + g * f$ and $D_\mu X \equiv \partial_\mu\hat{X} - i[\hat{A}_\mu, \hat{X}]$. These equations, which we will call θ-evolution equations, determine how the fields should change when θ is varied, in order to describe equivalent physics. Their integration provides the desired map between two descriptions with different values of θ, to which we will refer as the Seiberg-Witten map.

The fact that two apparently so different descriptions can be equivalent is certainly remarkable. The authors in [3] provided a direct check that this is indeed the case by showing that the effective action in one description is mapped to the effective action in the other by the Seiberg-Witten map. However, they worked in the approximation of slowly-varying fields, which consists of neglecting all terms of order ∂F (or $\partial^2 X$). This approximation was used at two different stages. First, when the effective action for the massless fields on the brane was taken to be the Dirac-Born-Infeld (DBI) action. Indeed, this action can be derived from string theory precisely by neglecting such terms. Second, this approximation was also used to simplify the Seiberg-Witten map considerably.

Although this check provides direct evidence in favour of the equivalence of the two descriptions, it would be desirable to have an exact proof. This would consist of three steps. First, one would have to determine the effective action in each description exactly, namely to all orders in α'. Second, one would have to integrate (1.8) also exactly, namely to all orders in θ. Third, one would have to substitute the change of variables in one action and see that the other one is recovered. Of course, this procedure is impossible to put into practice, but we will see that it is still possible to provide some evidence that the Seiberg-Witten map works non-perturbatively in θ.

The idea is as follows. If one exact effective action is mapped to the other by the Seiberg-Witten map, then a classical solution of one action should also be mapped to a

[2] The two equations (1.8) are simply the dimensional reduction of that for a ten-dimensional gauge field $\hat{A}_M$ $(M = 0, \ldots, 9)$, which was given in [3].

classical solution of the other. Of course, since we do not know the exact effective actions, we do not know any non-trivial exact solutions either, except for one case: the BIon [4, 5, 6]. This is a 1/2-supersymmetric solution of the ordinary DBI theory of a D-brane. In general, supersymmetric solutions of the worldvolume theories of branes have a natural interpretation as intersections of branes. The BIon is the prototype of this fact: it is the worldvolume realization of a fundamental string ending on a D-brane. What makes the BIon solution special is that, although originally discovered as a solution of the DBI action [4, 5], it has been shown to be a solution of the exact effective action to all orders in α' [7]. Perhaps this should not be surprising since, after all, the fact that a fundamental string can end on a D-brane is the defining property of D-branes [8].

When no background B-field is present, the string ends orthogonally on the brane (see figure 1(a)). When a constant *electric* [3] B-field is turned on, supersymmetry requires the string to tilt a certain angle γ determined by B (see figure 1(b)). This can be intuitively understood, because the background B-field induces a constant electric field on the brane [4]. Since the endpoint of the string is electrically charged, the string is now forced to tilt in order for its tension to compensate the electric force on its endpoint.

Our strategy will be to identify a BIon-like solution of the effective action in the non-commutative description with the value of θ determined by B as in (1.5). Since the exact effective action is not known, we will work with the lowest order approximation in α' (see (3.5)). We will then integrate the θ-evolution equations exactly to find what this non-commutative BIon is mapped to in the ordinary description. The result is that it is mapped to a tilted ordinary BIon, as described above, and that the tilting angle γ agrees precisely with the value determined by B [5].

Our result can be interpreted in two ways. On one hand, if one accepts that the equivalence between the ordinary and the non-commutative descriptions, as determined by the Seiberg-Witten map, is valid non-perturbatively in θ, then the result proves that the non-commutative BIon is a solution of the exact non-commutative effective action, namely it is a solution to all orders in α'. On the other hand, motivated by the defining property of D-branes mentioned above, one could directly conjecture that the non-commutative BIon is a solution of the exact non-commutative effective action. In this case, one could regard our result as evidence in favour of the exact equivalence of both descriptions beyond the perturbative level in θ. Whichever point of view one adopts, the result sheds some light on another related question raised in [3]: the convergence of the series in θ generated by the equations (1.8). As far as we are aware, this series has only been shown to converge for the case of constant field strength [3]. Our example shows that it converges in a less simple case.

In this paper we will concentrate on the case of D3-branes, but the analysis for BIons applies to D-branes of arbitrary dimension. The reason for this restriction is that we will briefly comment on solutions more general than BIons, namely on dyons. These are solutions of a D3-brane worldvolume theory carrying both electric and magnetic charges which constitute the worldvolume realization of (p, q)-strings ending on a D3-brane. They fill out

[3]An electric B-field leads, through (1.5), to an electric θ in the non-commutative theory. There is some controversy about whether or not such a theory makes sense. We will postpone the discussion of this issue until the last section.

[4]Fundamental strings ending on D-branes with constant electric fields on their worldvolumes have also been studied in [9]. We will clarify the relationship of the results in [9] with ours at the end of section 2.

[5]The fact that tilted brane configurations are related to monopoles and dyons in non-commutative gauge theories was first pointed out in [10] by working to first order in θ.

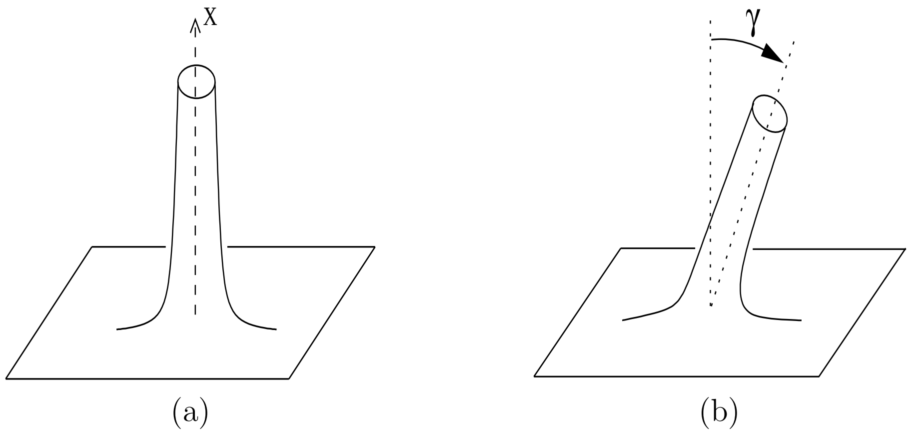

Figure 1: The worldvolume realization of a fundamental string ending on a D-brane: (a) in the absence of B-field, and (b) in the presence of an electric B-field.

orbits of the $SL(2,\mathbb{Z})$ duality group of type IIB string theory which, on the worldvolume of a D3-brane, becomes an electromagnetic duality group. In the last section we will briefly comment on the possible role of $SL(2,\mathbb{Z})$ in the non-commutative description of D3-branes. Section 2 is mainly a review. Sections 3 and 4 contain original results.

2 Ordinary BIons in a background electric B-field

In this section we will consider the ordinary description of the worldvolume theory of a D3-brane in the approximation of slowly-varying fields. The action is therefore the DBI action. We will begin by reviewing its dyonic solutions in the absence of a background B-field. Then we will concentrate on the case of purely electric solutions, the so-called 'BIons'. Finally we will see how the BIons are modified when a constant electric background B-field is turned on. In this section we will work in the static gauge $X^\mu \equiv \sigma^\mu$ $(\mu = 0, \ldots, 3)$. X^μ, together with the scalar fields on the brane X^i, $(i = 4, \ldots, 9)$, are target-space Cartesian coordinates, that is, the closed string metric g is assumed to take the form $g_{MN} = \eta_{MN}$ $(M, N = 0, \ldots 9)$ when expressed in these coordinates. Throughout this paper we will only allow for one scalar field to be excited. Therefore, we will consider the target-space to be effectively five-dimensional.

The dyonic solutions we wish to describe are the worldvolume realizations of (p, q)-strings ending orthogonally on the D3-brane. Since this is a spacetime-supersymmetric configuration, we have to look for a worldvolume-supersymmetric solution. Supersymmetric solutions of the D3-brane DBI action can be found either by imposing directly preservation of supersymmetry [4, 12] or by looking for BPS bounds on the energy [6]. In either case, the BPS equations for static 1/2-supersymmetric dyons carrying electric and magnetic charges p and q respectively, are [6, 4]

$$\boldsymbol{E} = \sin\alpha\, \boldsymbol{\nabla} X \,, \qquad \boldsymbol{B} = \cos\alpha\, \boldsymbol{\nabla} X \,, \tag{2.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields on the brane, $\tan\alpha = p/q$, and X is the only scalar field involved in the solution. The bound on the energy $\mathcal{E}$ which the solutions of (2.1) saturate is [6]

$$\mathcal{E} \geq \sqrt{Z_{el}^2 + Z_{mag}^2}\,, \tag{2.2}$$

where the electric and magnetic charges above are

$$Z_{el} = \int_\Sigma d^3\sigma\, \boldsymbol{E}\cdot\boldsymbol{\nabla}X\,, \qquad Z_{mag} = \int_\Sigma d^3\sigma\, \boldsymbol{B}\cdot\boldsymbol{\nabla}X\,, \tag{2.3}$$

and Σ is the D3-brane worldspace. Since both $\boldsymbol{E}$ and $\boldsymbol{B}$ are divergence free,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0\,, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, \tag{2.4}$$

(the former as a consequence of the Gauss law and the latter because of the Bianchi identity), these charges can be rewritten as surface integrals over the boundary of the brane worldspace:

$$Z_{el} = \int_{\partial\Sigma} d\mathbf{S}\cdot X\boldsymbol{E}\,, \qquad Z_{mag} = \int_{\partial\Sigma} d\mathbf{S}\cdot X\boldsymbol{B}\,. \tag{2.5}$$

This ensures that the charges are 'topological', in the sense that they only depend on the boundary conditions imposed on the fields. It is this topological nature which guarantees that the saturation of the bound automatically implies the equations of motion.

We see from (2.1) and (2.4) that X must be harmonic, that is, $\nabla^2 X = 0$. Given a harmonic function X, the electric and magnetic fields are determined by (2.1). A dyon is then associated with an isolated singularity of X.

We will concentrate for the rest of this section on the BIon. It corresponds to $\sin\alpha = \pm 1$ in (2.1), and therefore satisfies

$$\boldsymbol{E} = -\boldsymbol{\nabla}X\,, \tag{2.6}$$

where we have chosen the minus sign for convenience. The most general $SO(3)$-symmetric solution is then given (up to a gauge transformation) by

$$X = A_0 = \frac{e}{4\pi|\boldsymbol{\sigma}|}\,, \qquad \boldsymbol{A} = 0\,, \tag{2.7}$$

where $\boldsymbol{\sigma} = (\sigma^a)$, $a = 1, 2, 3$. It corresponds to a fundamental string ending orthogonally on the brane at $\boldsymbol{\sigma} = 0$, as depicted in figure 1(a). As mentioned above, (2.7) is a solution of (classical) string theory to all orders in α' [7], that is, when all corrections to the DBI action involving higher derivatives of the fields are taken into account.

Since the BIon (2.7) saturates the BPS bound (2.6), its energy equals its charge. Furthermore, the latter is easily calculated with the help of (2.5). The boundary of the BIon worldspace consists of a two-sphere at $|\boldsymbol{\sigma}| \to \infty$ and another at $|\boldsymbol{\sigma}| \to 0$. The surface integral over the former vanishes. Over the latter, it diverges. We can regularize it by integrating over a small two-sphere S_ϵ of radius ϵ. Since X is constant on this sphere we are left with

$$\mathcal{E} = |Z_{el}| = \lim_{\epsilon\to 0}\left|X(\epsilon)\int_{S_\epsilon} d\mathbf{S}\cdot\boldsymbol{E}\right| = e\lim_{\epsilon\to 0} X(\epsilon)\,. \tag{2.8}$$

This shows that the energy of the BIon is precisely the energy of an infinite string of constant tension [4, 6]. To compare with the BIon in the presence of a B-field and with the non-commutative BIon, it will be convenient for us to rewrite (2.8) as

$$\mathcal{E} = |Z_{el}| = z_{el}\lim_{\epsilon\to 0} I(\epsilon)\,, \tag{2.9}$$

where

$$z_{el} = e^2\,, \qquad I(\epsilon) = \frac{1}{4\pi\epsilon}\,. \tag{2.10}$$

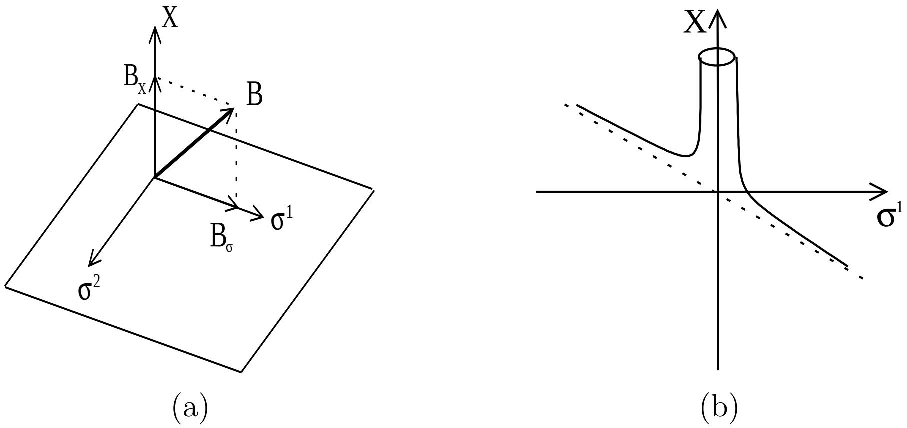

Figure 2: (a) Decomposition of the electric B-field, and (b) projection on the σ^1-X plane of the 'tilted' BIon (2.12).

Let us now see how the BIon solution is modified when a constant electric background B-field is turned on. We assume that the target-space ten-vector B_{0M} no longer vanishes (but we still impose the restriction that B has no magnetic components). In general, any non-vanishing component of B along directions 'transverse' to the brane can be gauged away. In our case, however, we have to be careful, because we are looking for a configuration in which one scalar field is excited. In other words, the worldspace of the brane is not a flat three-plane extending along the directions 1, 2 and 3. Therefore, we need to consider the component of B along the direction labelled by X (see figure 2(a)), to which we will refer as B_X. Without loss of generality, we can take the component of B along the 123-space to point along the 1-direction. We will denote this component by B_σ. The remaining components of B_{0M} can be gauged away and we will therefore set them to zero.

In the presence of the B-field, the BIon BPS equation which guarantees the preservation of some fraction of supersymmetry must be modified: the field strength $F = dA$ is replaced by $\mathcal{F} = F + B^\star$. This can be easily understood, since the supersymmetry condition has to be gauge-invariant under the two $U(1)$ gauge symmetries (1.1) and (1.2). Thus, (2.6) becomes

$$\mathcal{F}_{0a} = -\partial_a X \,, \tag{2.11}$$

whose solution is now

$$A_0 = \frac{e}{4\pi|\boldsymbol{\sigma}|} \,, \qquad \boldsymbol{A} = 0 \,, \qquad X = \frac{1}{1+B_X}\,\frac{e}{4\pi|\boldsymbol{\sigma}|} - \frac{B_\sigma}{1+B_X}\,\sigma^1 \,. \tag{2.12}$$

Note the appearance of a term linear in the worldspace coordinate σ^1 in the expression for the scalar field. This is the term responsible for the tilt of the string (see figure 2(b)). Indeed, although the spike coming out of the brane at $|\boldsymbol{\sigma}| = 0$ still points along the X-axis and the brane worldspace is still asymptotically flat, the latter no longer asymptotically extends along the 1-direction.

The energy of the solution (2.12) is computed analogously as we did with (2.7). The surface integral at infinity still vanishes (the term in the scalar field which is linear in σ^1 does not contribute because it changes sign under $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$). Thus, we obtain again (2.9), but with z_{el} now given by

$$z_{el} = \frac{e^2}{1+B_X} \tag{2.13}$$

We would like to close this section with some clarifications. The first one is that it might appear that (2.12) is not the most general solution of (2.11): we could have included terms linear in the worldspace coordinate both in X and in A_0, with appropriate coefficients such that (2.11) be satisfied. However, this apparently more general solution is related to (2.12) by a gauge transformation of the type (1.2); they are therefore physically equivalent. (Admittedly, such a transformation would shift the value of B, but this is allowed since B is generic in our analysis.) In particular, by choosing the target-space one-form as

$$\Lambda = \Big(B_\sigma \sigma^1 + B_X X \Big) \, d\sigma^0 \,, \tag{2.14}$$

the solution (2.12) in the presence of a non-vanishing B-field is mapped to a configuration with $A_0 = X$ (where X is still given by (2.12)) and vanishing B-field. This configuration satisfies (2.6), and is precisely of the form considered in [9]. These considerations therefore clarify the relation between the solution studied in [9] and the one we have presented: they are related by a gauge symmetry of the theory; whether the constant electric field on the brane is induced from the background or whether it arises from the worldvolume gauge field itself is a matter of gauge choice. We have chosen to work in a gauge in which $B \neq 0$ since it is in this case that a non-commutative alternative description exists.

The second remark is that the proof in [7] that the BIon (2.7) is an exact solution to all orders in α' assumed the absence of a background B-field. Therefore, one might question whether the result also holds for our BIon (2.12). The answer is affirmative, because, as we have just explained, (2.12) is related by a gauge symmetry of the theory to the configuration studied in [9] in which $B = 0$. As pointed out in [9], this latter configuration is indeed a solution to all orders in α', because it satisfies (2.6), which was the only assumption in [7] (as opposed to any assumption concerning the specific form of the solution, such as (2.7)).

3 Non-commutative D3-brane Dyons

In this section we will first establish the non-commutative version of the BPS equations (2.1). Then we will construct an exact solution for the case of purely electric charge, which we will refer to as the 'non-commutative BIon'. In the next section, we will show that this solution is mapped to the ordinary BIon (2.12) in a B-field by the Seiberg-Witten map. Therefore, the metric we must use here is the open string metric G as determined by g and B in (1.5). This is important since, although both g and G are flat, they are not necessarily simultaneously diagonal in the same coordinate system. In this section it will be convenient to work in a system in which G takes the simplest form $G_{MN} = \eta_{MN}$ (note that we are referring to the target-space coordinate system, and that therefore it includes the scalar field on the brane). The system in which G takes this form is obtained as follows (see figure 3). Let us take the first worldspace coordinate ξ^1 along the direction of B, and the scalar field Y to be orthogonal to it. In this system we have

$$B = b \, d\xi^0 \wedge d\xi^1 \,, \tag{3.1}$$

where b is a positive constant, but G does not yet take the desired form. However, the simple rescaling

$$\rho^0 = \sqrt{1-b^2} \, \xi^0 \,, \qquad \rho^1 = \sqrt{1-b^2} \, \xi^1 \tag{3.2}$$

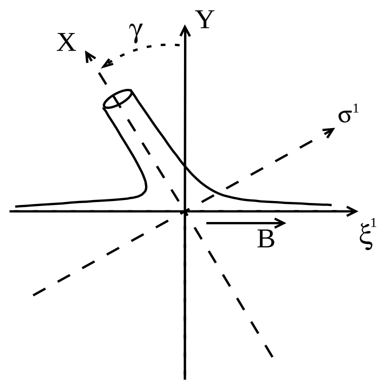

Figure 3: The convenient coordinate system in the analysis of the non-commutative BIon, and the projection to the ξ^1-Y plane of the solution (4.12).

brings G into such a form:

$$ds^2_{(G)} = -d\rho_0^2 + d\rho_1^2 + d\rho_2^2 + d\rho_3^2 + dY^2 \tag{3.3}$$

In this coordinate system θ also takes a very simple form: its only non-vanishing components are

$$\theta^{01} = -\theta^{10} = b\,. \tag{3.4}$$

After these preliminaries, we are ready to write down the non-commutative D3-brane worldvolume action. We will take it to be the lowest order approximation in α', namely

$$S = \int d^{1+3}\rho \left(-\frac{1}{4}\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} - \frac{1}{2}D_\mu\hat{Y} * D^\mu\hat{Y}\right)\,. \tag{3.5}$$

All indices above are contracted with G.

One might think that a Hamiltonian analysis of the non-commutative action is required in order to obtain its energy functional, which we certainly need to derive a BPS bound on the energy. This seems difficult in view of the non-local nature of the action. (Note that this non-locality is not only in space but also in time, since we will not impose the restriction $\theta^{0a} = 0$.) The problem can be circumvented by noting that, although the presence of θ breaks Lorentz invariance, the theory is still translation-invariant. Therefore we can obtain the Lagrangian energy $\mathcal{E}$ as the conserved quantity under time translations. The result is

$$\mathcal{E} = \int_\Sigma d^3\rho \left(\frac{1}{2}\hat{\boldsymbol{E}}^2 + \frac{1}{2}\hat{\boldsymbol{B}}^2 + \frac{1}{2}(D_0\hat{Y})^2 + \frac{1}{2}(\boldsymbol{D}\hat{Y})^2\right)\,, \tag{3.6}$$

where $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ are the non-commutative electric and magnetic fields, that is,

$$\hat{E}_a = \hat{F}_{0a}\,, \qquad \hat{B}_a = \frac{1}{2}\epsilon_{abc}\hat{F}_{bc}\,. \tag{3.7}$$

For covariantly static configurations, namely with $D_0\hat{Y} = 0$, the energy can be rewritten as

$$\mathcal{E} = \int_\Sigma d^3\rho \left[\frac{1}{2}\left(\hat{\boldsymbol{E}} - \sin\alpha\boldsymbol{D}\hat{Y}\right)^2 + \frac{1}{2}\left(\hat{\boldsymbol{B}} - \cos\alpha\boldsymbol{D}\hat{Y}\right)^2 + \sin\alpha\hat{\boldsymbol{E}}\cdot\boldsymbol{D}\hat{Y} + \cos\alpha\hat{\boldsymbol{B}}\cdot\boldsymbol{D}\hat{Y}\right]\,, \tag{3.8}$$

from which the desired BPS bound follows immediately:

$$\mathcal{E} \geq \sqrt{Z_{el}^2 + Z_{mag}^2}\,. \tag{3.9}$$

The electric and magnetic charges above are the natural non-commutative generalizations of (2.3):

$$Z_{el} = \int_\Sigma d^3\sigma\, \hat{\boldsymbol{E}} \cdot \boldsymbol{D}\hat{Y}\,, \qquad Z_{mag} = \int_\Sigma d^3\sigma\, \hat{\boldsymbol{B}} \cdot \boldsymbol{D}\hat{Y}\,. \tag{3.10}$$

The bound is saturated precisely when the non-commutative BPS equations

$$\hat{\boldsymbol{E}} = \sin\alpha\, \boldsymbol{D}\hat{Y}\,, \qquad \hat{\boldsymbol{B}} = \cos\alpha\, \boldsymbol{D}\hat{Y} \tag{3.11}$$

hold [6]. They are also the natural generalizations of their commutative counterparts (2.1).

As well as the BPS equations, the Gauss law and the Bianchi identity (2.4) should also be promoted to their non-commutative versions

$$\boldsymbol{D} \cdot \hat{\boldsymbol{E}} = 0\,, \qquad \boldsymbol{D} \cdot \hat{\boldsymbol{B}} = 0\,. \tag{3.12}$$

A number of comments are in order here. The new Bianchi identity follows immediately from the definition of $\hat{F}$. However, it is not clear to us whether it constitutes a locally sufficient integrability condition (as it does in an ordinary gauge theory) which ensures that $\hat{F}$ can be written as the covariant derivative of a gauge potential. The Gauss law above is nothing else than one of the equations of motion. However, a rigorous proof that it is a constraint in the non-commutative theory would require a careful analysis, which is difficult again due to its non-locality. Nevertheless, there are two observations which are worth noticing. First of all, at least in the case of purely magnetic θ, that is, when $\theta^{0a} = 0$, the non-commutative theory becomes local in time and the Hamiltonian analysis is straightforward [14]. In this case one can prove that the Gauss law (which contains only first order time derivatives in its Lagrangian form) becomes a Hamiltonian constraint. Second, both the Bianchi identity and the Gauss law are required in order to rewrite the electric and magnetic charges as surface integrals:

$$Z_{el} = \int_{\partial\Sigma} d\mathbf{S} \cdot \hat{Y}\hat{\boldsymbol{E}}\,, \qquad Z_{mag} = \int_{\partial\Sigma} d\mathbf{S} \cdot \hat{Y}\hat{\boldsymbol{B}}\,. \tag{3.13}$$

As in the ordinary case, this is what ensures that they have a topological nature, which in turn guarantees that the saturation of the bound automatically implies the equations of motion.

Now we are ready to finally write down the non-commutative BIon solution. It is simple to check that the configuration

$$\hat{A}_0^{(\rho)} = \hat{Y} = \frac{q}{4\pi|\boldsymbol{\rho}|}\,, \qquad \hat{\boldsymbol{A}}^{(\rho)} = 0\,, \tag{3.14}$$

solves both the non-commutative BPS equations (3.11) (with $\sin\alpha = -1$ as before) and the non-commutative Gauss law (3.12). In (3.14) we have $\boldsymbol{\rho} = (\rho^a)$, and the superscript

[6]The case with $\cos\alpha = 1$ in (3.11), which corresponds to a monopole, was studied in [11] in the context of a $U(2)$ gauge group. The non-commutative monopole equation was solved to first order in θ. The solution exhibits a certain non-locality corresponding to the tilt of the D-string ending on the brane.

(ρ) denotes that the above are the components of the gauge potential one-form in the ρ-coordinate system, namely that $\hat{A} = \hat{A}_0^{(\rho)}\, d\rho^0$. Its components in the ξ-coordinate system are, by virtue of (3.2),

$$\hat{A}_0^{(\xi)} = \sqrt{1-b^2}\, \hat{A}_0^{(\rho)}\,, \qquad \hat{\boldsymbol{A}}^{(\xi)} = 0\,. \tag{3.15}$$

Hereafter we will denote $\hat{A}_0^{(\rho)}$ simply by $\hat{A}_0$ until the moment in which we have to compare with (2.12).

The solution (3.14) is the announced non-commutative BIon. Its charge is again easily computed as we did in the previous section. The result is again (2.9) with

$$z_{el} = q^2\,. \tag{3.16}$$

and we see from (2.13) and (3.16) that we must choose

$$q = \frac{e}{\sqrt{1+B_X}} \tag{3.17}$$

in order for the charges (or equivalently, for the energies) of the non-commutative and of the ordinary BIon to agree.

4 From Non-commutative to Commutative BIons

Our goal in this section is to show that the solution (3.14) of the non-commutative theory (with non-commutativity parameter θ determined by B as in (3.4)) is precisely mapped to the ordinary BIon (2.12) in the presence of the B-field. We should therefore integrate the θ-evolution equations (1.8) exactly, with the initial conditions (3.14) for the fields at the initial value of θ given in (3.4). However, there is an *a priori* obstacle for doing so which is worth discussing. Indeed, the θ-evolution equations constitute a system of coupled partial differential equations, whose (local) integrability conditions are that the crossed derivatives be equal, namely that

$$\frac{\partial^2 \hat{A}_\mu}{\partial\theta^{\gamma\rho}\partial\theta^{\alpha\beta}} = \frac{\partial^2 \hat{A}_\mu}{\partial\theta^{\alpha\beta}\partial\theta^{\gamma\rho}}\,, \tag{4.1}$$

and similarly for the scalar fields. As proved in [13] [7], these conditions are in general not satisfied. This means that the evolution of the fields in 'θ-space' determined by integrating the equations (1.8) between some initial and final values θ_0 and θ_1 depends on the path followed from θ_0 to θ_1. The choice of path should be made according to some physical input [8].

In our case, we are interested in the θ-evolution of the BIon (3.14) starting from an initial θ which is purely electric, and ending at $\theta = 0$. This suggests that we should restrict ourselves to the hyperplane $\theta^{ab} = 0$ in θ-space, and consider paths contained within this hyperplane. This restriction can be physically further motivated as follows. Suppose that we start with a D3-brane in the absence of any B-field. In this situation only the ordinary description is available. Now we smoothly turn on the electric components of B. As soon as B is different from zero, both the ordinary and the non-commutative descriptions are

[7] I thank Joan Simón for drawing my attention to this reference.

[8] As explained in [13] there is no path dependence for the case of a $U(1)$ gauge group if terms of $\mathcal{O}(\partial\hat{F})$ are neglected.

available. It follows from its definition (1.5) that, in the latter description, θ will also be purely electric. We can now imagine following the evolution of the fields as a function of θ (or, equivalently, of B) as it increases. We see that in this physical situation we are following a path which lies in the $\theta^{ab} = 0$ hyperplane.

The restriction $\theta^{ab} = 0$ simplifies the θ-evolution equations dramatically, because it implies that the $*$-product of any two time-independent functions f and g reduces to their ordinary product, namely that $f * g = fg$. With this simplification the equations (1.8) become

$$\begin{aligned} \frac{\partial \hat{A}_0}{\partial \theta^b} &= -\hat{A}_0\, \partial_b \hat{A}_0 && (4.2) \\ \frac{\partial \hat{A}_a}{\partial \theta^b} &= -\hat{A}_0 \partial_b \hat{A}_a + \frac{1}{2}\hat{A}_0 \partial_a \hat{A}_b - \frac{1}{2}\hat{A}_b \partial_a \hat{A}_0 \,, && (4.3) \\ \frac{\partial \hat{Y}}{\partial \theta^b} &= -\hat{A}_0\, \partial_b \hat{Y} \,, && (4.4) \end{aligned}$$

where $\theta^a \equiv \theta^{0a}$. Note that in the above equations $\partial_b \equiv \partial/\partial\rho^b$.

The unique solution of (4.3) with the initial condition that $\hat{A}_a$ vanishes at some initial value of θ is that it vanishes for all values of θ, regardless of the chosen path. Furthermore, taking the derivative of (4.2) and using (4.2) itself, one can easily check that the crossed derivatives of $\hat{A}_0$ coincide. Finally, (4.2) and (4.4), together with the initial condition $\hat{A}_0 = \hat{Y}$, show that $\hat{A}_0$ and $\hat{Y}$ remain equal for all values of θ. We thus conclude that, in the hyperplane $\theta^{ab} = 0$, the θ-evolution of the BIon (3.14) is path-independent, and that to determine it we need only solve (4.2) with the initial condition

$$\hat{A}_0(\boldsymbol{\rho}, \boldsymbol{\theta})\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \frac{q}{4\pi|\boldsymbol{\rho}|}\,, \tag{4.5}$$

where $\boldsymbol{\theta} = (\theta^a)$ and $\boldsymbol{\theta}_0 = (b, 0, 0)$.

(4.2) constitutes a system of three quasi-linear partial differential equations which can be solved by the method of characteristics:[9] instead of solving (4.2) directly, one introduces a new three-vector $\boldsymbol{t} = (t^a)$ and solves

$$\frac{\partial \rho^b}{\partial t^a} = \delta^b_a\, \hat{A}_0\,, \qquad \frac{\partial \theta^b}{\partial t^a} = \delta^b_a\,, \qquad \frac{\partial \hat{A}_0}{\partial t^a} = 0 \tag{4.6}$$

with initial conditions that depend on a further three-vector $\boldsymbol{s} = (s^a)$, namely

$$\boldsymbol{\rho} = \boldsymbol{s}\,, \qquad \boldsymbol{\theta} = \boldsymbol{\theta}_0\,, \qquad \hat{A}_0 = \frac{q}{4\pi|\boldsymbol{s}|} \qquad\qquad \text{at } \boldsymbol{t} = 0\,. \tag{4.7}$$

The physical interpretation of both $\boldsymbol{t}$ and $\boldsymbol{s}$ will become clear shortly.

(4.6) and (4.7) determine $\boldsymbol{\rho}$, $\boldsymbol{\theta}$ and $\hat{A}_0$ as functions of $\boldsymbol{t}$ and $\boldsymbol{s}$. It is easy to see that the solution is

$$\boldsymbol{\rho}(\boldsymbol{t}, \boldsymbol{s}) = \frac{q}{4\pi|\boldsymbol{s}|}\boldsymbol{t} + \boldsymbol{s}\,, \qquad \boldsymbol{\theta}(\boldsymbol{t}, \boldsymbol{s}) = \boldsymbol{t} + \boldsymbol{\theta}_0\,, \qquad \hat{A}_0(\boldsymbol{t}, \boldsymbol{s}) = \frac{q}{4\pi|\boldsymbol{s}|}\,. \tag{4.8}$$

The (local) inversion of the first two relations above would determine $\boldsymbol{t}(\boldsymbol{\rho}, \boldsymbol{\theta})$ and $\boldsymbol{s}(\boldsymbol{\rho}, \boldsymbol{\theta})$, which could be substituted into the third one to obtain the desired solution $\hat{A}_0(\boldsymbol{\rho}, \boldsymbol{\theta})$. To

[9] I am grateful to Emili Elizalde for help at this point.

see that the function $\hat{A}_0$ so determined satisfies (4.2) we simply have to apply the chain rule and use (4.6) to obtain:

$$0 = \frac{\partial \hat{A}_0}{\partial t^a} = \frac{\partial \hat{A}_0}{\partial \theta^b}\frac{\partial \theta^b}{\partial t^a} + \frac{\partial \hat{A}_0}{\partial \rho^b}\frac{\partial \rho^b}{\partial t^a} = \frac{\partial \hat{A}_0}{\partial \theta^a} + \hat{A}_0\,\frac{\partial \hat{A}_0}{\partial \rho^a} \tag{4.9}$$

It is also obvious from (4.7) that $\hat{A}_0$ satisfies the required initial condition (4.5).

We see from (4.8) how to interpret $\boldsymbol{t}$: it is simply the difference between the initial and the final values of the non-commutativity parameter. Therefore, in our case we will be interested in examining the values of the fields at

$$\boldsymbol{t} = (-b, 0, 0)\,. \tag{4.10}$$

(4.8) also shows that $\boldsymbol{s} = (s^a)$ are nothing else than intrinsic coordinates on the D3-brane worldspace. Indeed, recall that we argued that the scalar $\hat{Y}$ and the potential $\hat{A}_0$ coincide for all values of θ. Therefore, for fixed $\boldsymbol{t}$, (4.8) determines the (static) embedding of the brane in spacetime as

$$\boldsymbol{s} \mapsto \Big(\boldsymbol{\rho}(\boldsymbol{s}), \hat{Y}(\boldsymbol{s})\Big) \tag{4.11}$$

So we see that (4.8) provides the solution (namely the gauge potential and the embedding of the brane in target-space) in parametric form.

In order to check that the solution (4.8) at $\theta = 0$ is, as we claim, precisely the same as (2.12), we have to express (4.8) in the same coordinate system as (2.12).

First, we have to undo the rescaling (3.2), because the closed string metric g takes the Minkowski form in the ξ-coordinates, but not in the ρ-coordinates. We thus have, using (3.2), (3.15), (4.8) and (4.10), that, at $\theta = 0$:

$$\xi^1 = -\frac{b}{\sqrt{1-b^2}}\,\frac{q}{4\pi|\boldsymbol{s}|} + \frac{1}{\sqrt{1-b^2}}\,s^1\,, \qquad Y = \frac{q}{4\pi|\boldsymbol{s}|}\,, \tag{4.12}$$

$$A_0 = \sqrt{1-b^2}\,\frac{q}{4\pi|\boldsymbol{s}|}\,, \qquad \boldsymbol{A} = 0\,. \tag{4.13}$$

Note that we have dropped the 'hats' on the fields, because the above are already their values in the ordinary description. Note also that, although we did not write the superscript (ξ) explicitly, A_0 and $\boldsymbol{A}$ now denote the components of the gauge potential in the ξ-coordinate system.

We have drawn the solution (4.12) (which is still expressed in parametric form) in figure 3. For $|\boldsymbol{s}| \to \infty$ we have $Y \sim 0$. Therefore the brane is asymptotically flat and extends along the directions labeled by ξ^1, ξ^2 and ξ^3, as shown in the figure. On the contrary, for $|\boldsymbol{s}| \to 0$, we see that

$$Y \sim -\frac{\sqrt{1-b^2}}{b}\,\xi^1 \tag{4.14}$$

This means that the spike comes out of the region $|\boldsymbol{\xi}| \to 0$ in a direction at an angle γ with the Y-axis (see figure 3 again), where

$$\tan\gamma = \frac{b}{\sqrt{1-b^2}} \tag{4.15}$$

We have labelled this direction by X, and the orthogonal direction in the ξ^1-Y plane by σ^1. Recall that the ordinary BIon solution (2.12) is written in the static gauge and in a

coordinate system in which the spike points along the transverse scalar field. In our case, this happens precisely in the σ-X coordinate system. Therefore, the last step we have to take is to rotate the solution (4.12) by an angle γ. Defining

$$\begin{pmatrix} \sigma^1 \\ X \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \xi^1 \\ Y \end{pmatrix} \tag{4.16}$$

we find that $\sigma^1 = s^1$ and that

$$X = \frac{1}{\sqrt{1-b^2}} \frac{q}{4\pi|\boldsymbol{s}|} - \frac{b}{\sqrt{1-b^2}}\, s^1 \,. \tag{4.17}$$

Finally, we see from (3.1) and (4.16) that

$$B_\sigma = b\sqrt{1-b^2}\,, \qquad B_X = -b^2\,. \tag{4.18}$$

Therefore, using (3.17), we arrive at the final form of our solution, expressed in the static gauge:

$$A_0 = \frac{e}{4\pi|\boldsymbol{\sigma}|}\,, \qquad X = \frac{1}{1+B_X}\frac{e}{4\pi|\boldsymbol{\sigma}|} - \frac{B_\sigma}{1+B_X}\sigma_1\,. \tag{4.19}$$

It coincides precisely with (2.12), as we claimed.

5 Discussion

In this last section we wish to address a number of issues which were not fully discussed in the text.

The first one concerns the interpretation of the scalar fields in the non-commutative theory. In the ordinary description of a single D-brane, the scalars unambigously determine the embedding of the brane in spacetime. This is the reason why many phenomena in field theories acquire a clearer geometrical interpretation when such theories are realized as worldvolume theories of branes. In the non-commutative description, this interpretation is much less clear. The reason is that the scalar fields, even in the case of one single D-brane, are no longer gauge-invariant but gauge-covariant quantities; namely they transform as $\delta\hat{X} = i[\hat{\lambda}, \hat{X}]$. This problem is related to the fact that all gauge-invariant quantities in non-commutative gauge theories seem to be non-local, obtained after integrating some gauge-covariant quantity. Presumably, one can determine global properties of the brane embedding from the non-commutative scalar fields, such as winding numbers, etc., by integrating appropriate expressions, but not the local details of the embedding. The reason why we did not have to resolve this problem in our discussion of the non-commutative BIon is that we were only interested in identifying a solution of the non-commutative theory which was exactly mapped to the ordinary BIon in the presence of a B-field. Since the Seiberg-Witten map maps gauge orbits into gauge orbits and the ordinary scalar fields are gauge-invariant, any scalar field configuration in the non-commutative theory which is gauge-equivalent to (3.14) would have also been mapped to the ordinary BIon. We simply chose the simplest representative of $\hat{Y}$ in its gauge-equivalence class.

The second point we wish to discuss here is whether it makes sense to consider a non-commutativity matrix with non-vanishing electric components [10]. In the ordinary description, it certainly makes sense to consider $B_{0a} \neq 0$, which leads through (1.5) to $\theta^{0a} \neq 0$.

[10] I would like to thank Michael B. Green and Paul K. Townsend for a conversation on this point.

Moreover, even if in one coordinate system B has no electric components, we can always choose to describe the physics in a frame in which B does have electric components. The only question is whether the equivalence between the ordinary and the non-commutative descriptions still holds *in this frame*. It seems that our result can be regarded as evidence in favour of the affirmative answer to this question.

We would like to close this paper with a little digression on the role of S-duality in the non-commutative theory, and more generally of the full $SL(2,\mathbb{Z})$ duality group of type IIB string theory. Consider the BIon solution in the ordinary description of D3-branes and in the absence of any B-field. As we have explained, this is the worldvolume realization of the spacetime configuration in which a fundamental string ends on the D3-brane. The S-duality of string theory maps this configuration into one in which a D-string ends on the D3-brane. Thus, from the worldvolume point of view of the latter, S-duality is an inherited symmetry which corresponds to electromagnetic duality. It maps the BIon into the monopole. These two objects are therefore equivalent, in the sense that they are related by a symmetry of the theory. One might think that this is also the case in the non-commutative worldvolume description, perhaps with a further exchange of the electric and the magnetic components of θ. However, this is not true. The reason is that a BIon in the non-commutative theory corresponds, as we have seen, to a fundamental string ending on the D3-brane in the presence of a B-field. S-duality maps this configuration into a D-string ending on the D3-brane in the presence now of a *Ramond-Ramond* C-field. Clearly, this does not correspond to a monopole in the non-commutative theory, which should instead correspond to a D-string ending on the D3 in the presence of a B-field. These considerations raise the following interesting question. There exists an $SL(2,\mathbb{Z})$-covariant worldvolume action for D3-branes coupled to a supergravity background [15], which can consist, in particular, of a flat background with constant B and C fields. The case of vanishing C-field, for which we know that a non-commutative description is possible, is mapped to the generic one by $SL(2,\mathbb{Z})$. Since $SL(2,\mathbb{Z})$ is a symmetry of IIB string theory, should there not exist an $SL(2,\mathbb{Z})$-covariant non-commutative description of D3-branes in the presence of both constant B and C fields?

Note added: While this paper was being written, I learned about [16], which has some overlap with section 3.

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